COSMOLOGY
Problems 1
22 January, 2010

1. We define the Hubble Constant, H_0 by the relationship

$$cz = H_0 d$$

where c is the speed of light and d is the distance to the object.

   (a) Show that $H_0 = 72$ km/s/Mpc makes dimensional sense.
   (b) Simplify this value of H_0 so that it has SI units.
   (c) If the universe had been expanding at this rate since time $t = 0$ (i.e., if the value of $\dot{a}$ were constant), what would the age of the Universe be today?

2. The two Friedmann Equations

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3} - \frac{kc^2}{a^2}$$

and

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + 3p/c^2\right)$$

are not independent of the fluid conservation equation

$$\frac{d\rho}{dt} + 3\left(\rho + p/c^2\right)\frac{\dot{a}}{a} = 0\ .$$

Prove this for the case of a flat Universe and pressureless matter ($p = 0$).

3. The critical density of an FRW universe is

$$\rho_c = \frac{3H_0^2}{8\pi G}\ .$$

   (a) Using H_0 from above calculate the critical density in SI units and also in hydrogen atoms per cubic centimetre.
   (b) Estimate the mass density of galaxies in the Universe in SI units, given a typical separation of 1 Mpc and a typical mass of $10^{11}M_\odot$. Compare this with the critical density of an FRW Universe.

4. In the lectures, we examined solutions for the expansion when the Universe contained either dust ($p = 0$) or radiation $p = \rho c^2/3$. Suppose we have a more general equation of state $p = w\rho c^2$ where w is a constant in the range $-1 \le w \le 1$. Find solutions for $\rho(a)$, $a(t)$ and hence $\rho(t)$ for Universes containing such matter. Assume $k = 0$ in the Friedmann equation.

   What is the solution if $p = -\rho c^2$?

5. The full first-order Friedmann equation is given above. Consider the case $k > 0$, with a Universe containing only dust ($p = 0$), so that $\rho = \rho_0 a_0^3/a^3$. Demonstrate that the parametric solution

$$a(\theta) = \frac{a_0\Omega_0}{2(\Omega_0 - 1)}(1 - \cos\theta)\ , \qquad t(\theta) = \frac{H_0^{-1}\Omega_0}{2(\Omega_0 - 1)^{3/2}}(\theta - \sin\theta)\ ,$$

solves the equation, where $0 \le \theta < 2\pi$. Combine these plots to sketch $a(t)$. Sketch a and t as functions of θ and describe in qualitative terms the behaviour of the Universe. Combine these plots to sketch $a(t)$.

   Speculate on how these equations might change if $k < 0$ (consider what happens to trigonometric functions when you change from circles to hyperbolas.)

COSMOLOGY
Problems 2 **© A. Jaffe**
29 January, 2010

1. In lecture we always define our "comoving observers" not moving with respect to the expanding Universe. But real galaxies do have "extra" motion superimposed on the smooth expansion, called the "peculiar velocity". Write down an expression for the measured redshift that modifies Hubble's law to take this into account. Suppose that the typical RMS velocity dispersion of a galaxy is $\langle v_{\rm pec}^2\rangle^{1/2} = 750$ km/s. At what distance does the redshift give an estimate of the expansion velocity which is accurate to 5% or better?

2. Consider a universe with $k > 0$ (spherical geometry) and a nonzero cosmological constant $\Lambda > 0$. Using both the first- and second-order Friedmann Equations, show that we can satisfy both $\dot{a} = 0$ and $\ddot{a} = 0$ for all time. This is Einstein's original static model. If we fix the scale factor to be a constant $a = a_\Lambda$ in this model, what are the values of the density and cosmological constant required?

3. Although we won't be using it much in this course, a conventional way to describe the contents of the Universe used to be the deceleration parameter, q_0, defined by

$$q_0 = -\frac{a\ddot{a}}{\dot{a}^2} \ .$$

Show that in a matter-dominated Universe, $q_0 = \Omega_m/2$, and in a radiation-dominated Universe, $q_0 = \Omega_r$.

4. For a general equation of state ($w = p/\rho c^2$), identify a sufficient and necessary condition that $q_0 < 0$.

5. (a) Above what temperature is the mean energy of a photon in a thermal distribution sufficient to ionize hydrogen atoms? At what redshift did the radiation in the Universe have this temperature? Was the Universe matter- or radiation-dominated at this time?

   (b) Above what temperature is the mean energy of a photon in a thermal distribution sufficient to disassociate a Helium nucleus (very roughly — please don't try to calculate the exact energy required)? At what redshift did the radiation in the Universe have this temperature? Was the Universe matter- or radiation-dominated at this time?

COSMOLOGY
Problems 3 **© A. Jaffe**
9 February, 2010

1. The galaxy's age can be estimated by radioactive decay of Uranium. Uranium is produced as an r-process element in supernovae, and on this basis the initial abundances of U^{235} and U^{238} are expected to be:
$$\left.\frac{U^{235}}{U^{238}}\right|_{\text{initial}} = 1.65$$
The decay rates are
$$\begin{aligned}\lambda(U^{235}) &= 0.97\times 10^{-9}\ \text{yr}^{-1}\\ \lambda(U^{238}) &= 0.17\times 10^{-9}\ \text{yr}^{-1}\end{aligned}$$
Finally, the present abundance ratio is
$$\left.\frac{U^{235}}{U^{238}}\right|_{\text{final}} = 0.0072$$
Use the decay law $U(t) = U(0)\exp(-\lambda t)$ to estimate the age of the galaxy. Assuming the galaxy took a minimum of an additional 10^9 yr to form in the first place, obtain an upper limit on the value of the Hubble parameter h assuming a flat matter-dominated Universe (2001 exam, Q2). (Recall $H_0 = 100h$ km/s/Mpc)

2. Show that we can ignore the contribution of the radiation-dominated era to the present-day age of the Universe.

3. Expand the expressions for the angular diameter distance and luminosity distance in a Taylor's series around $z = 0$, and express the first two terms using H_0 and q_0 (from the last problem sheet).

4. Work out the angle subtended by an object of physical size l as a function of redshift in a flat, matter-dominated Universe (Einstein-de Sitter). Find the behaviour in the limits of small and large z, and provide a physical explanation. Show that the object appears the smallest if it is located at redshift $z = 5/4$.

COSMOLOGY
Problems 4
16 February, 2010

1. Show that the substitution $\nu' = \nu/(1+z)$ into the expression for the energy density of blackbody radiation as a function of ν results in a blackbody spectrum corresponding to a temperature $T = T/(1+z)$. Give an interpretation of the result.

2. Show that we can write the baryon-to-photon ratio $\eta = n_B/n_\gamma$ as

$$\eta = x\Omega_B h^2 (T/T_0)^\alpha$$

   with $T_0 = 2.73$ K and find the constant of proportionality x and the exponent α.

3. The entropy per degree of freedom of a particle species at temperature T is given by

$$s = \frac{2k\pi^2}{45} g\alpha (kT)^3$$

   where k is Boltzmann's constant, $\alpha = 1$ for bosons and $\alpha = 7/8$ for fermions. Photons have $g = 2$ degrees of freedom (two possible polarizations) and so do electrons and positrons (two spin states for each).

   For high enough temperatures, the photons are in thermal equilibrium with a plasma of electrons and positrons. When the Universe cools the electrons and positrons annihilate into photons. Assuming this process happens at constant entropy, so that $s(g_i, T_i) = s(g_f, T_f)$ where i and f correspond to before (initial) and after (final) the electrons and positrons annihilate, how much are the photons heated compared to the neutrinos (which are not heated, as they had already decoupled from the photons as discussed in class)? Given that the cosmic microwave background temperature is 2.728K, what temperature do you predict the cosmic neutrino background to be at? (*Hints: recall our definition of the effective degrees of freedom, g_*.*)

4. A neutrino with mass 10 eV is a dark matter candidate. At what temperature would the thermal energy of such a neutrino equal its mass energy?

   Before that epoch the neutrinos would be relativistic. Making the appropriate assumption as to whether the Universe is matter or radiation dominated, estimate how far the neutrinos would be able to travel up to that epoch. Assume they travel at the speed of light while relativistic.

   Since that time, the neutrinos have been non-relativistic and so have had negligible velocity, but the Universe has still been expanding. What is the factor of expansion, and what is the distance in the Universe as measured today that the neutrinos would have been able to travel? How does this scale compare to the scale of a typical structure in the Universe?

COSMOLOGY
Problems 5
23 February, 2010

1. Computer problems:

- Write a computer program to calculate our function

$$E(z) = \sqrt{\Omega_m(1+z)^3 + \Omega_r(1+z)^4 + \Omega_\Lambda + \Omega_k(1+z)^2}$$

with $\Omega_k = 1 - \Omega_m - \Omega_r - \Omega_\Lambda$ and plot the value of $\dot{a}/a$ as a function of z for a few different cases (such as the "concordance model" with $\Omega_k = 0$, $\Omega_m \simeq 0.3$, $\Omega_\Lambda \simeq 0.7$).

- Use this to calculate $\Omega_i(z)$.
- How would you have to change this to take into account the fact that more species of particles are relativistic at early times? Another way to think of this is to ask how you would take into account a varying equation of state, $w(z)$, for some species.
- If the Universe is *not* flat, but instead has only matter, at the value observed today, work out the value $\Omega_m(z)$ at the epoch of decoupling, $z_{\text{dec}} \simeq 1100$, at the epoch of matter-radiation equality (i.e., at the redshift that it would occur in our Universe) and at the $z \sim 10^6$. Repeat the calculation for a Universe with radiation as observed today (ignoring the additional relativistic species from the last question). We will discuss in class why this result, too, points to an open question in the big bang.
- (This is a bit more challenging as it requires numerical integration.) Modify your program to calculate the quantities $d_H(z)$, $d_A(z)$, $d_L(z)$ and the age of the Universe.
- Use these programs to work out the angular size of the horizon distance at the last scattering surface. We will discuss in class why this result, too, points to an open question in the big bang.

COSMOLOGY
Problems 6 **© A. Jaffe**
9 March, 2010

1. (Liddle 13.5-6)

   (a) Magnetic monopoles behave as non-relativistic matter. Suppose that at a Temperature corresponding to the GUT era, about $T_{GUT} = 3 \times 10^{28}$ K, magnetic monopoles were created with a density at that time of $\Omega_{\text{mon}}(T_{GUT}) = 10^{-10}$. Assuming the Universe has the critical density overall and is radiation dominated, what was the Temperature when the density of monopoles equalled that of radiation?

   (b) In the present Universe, $T \simeq 3$ K. Compute the value $\Omega_{\text{mon}}/\Omega_r$ would have at the present day. Is this compatible with observations?

   (c) If we have a period of inflation, the monopole density reduces as $1/a^3$ but the total density, dominated by the cosmological constant, remains fixed. Since that density will be converted to radiation after inflation, we can imagine that the radiation density remains constant during inflation. How much inflationary expansion is necessary so that the present density of monopoles matches that of radiation?

2. We saw in class that we could write the pressure and density of a scalar field as

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi) \qquad p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi)\ .$$

   (a) Using the fluid equation,

$$\dot{\rho} + 3H(\rho + p) = 0$$

   show that the equation of motion of the scalar field is

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0\ .$$

   (b) In the slow-roll approximation, we ignore the terms $\ddot{\phi}$ and $\dot{\phi}^2$.

   Use this to show that the Friedmann Equation and the equation of motion (together known as the *slow-roll equations* in this context) are therefore

$$H^2 \approx \frac{8\pi G}{3}V \qquad 3H\dot{\phi} \approx -V'(\phi)\ .$$

   (c) Show that the "number of e-foldings" is given by

$$N = \ln\left(\frac{a_2}{a_1}\right) = \int H\ dt = -8\pi G \int_{\phi_1}^{\phi_2} \frac{V(\phi)}{V'(\phi)}\ d\phi$$

   (d) One model for inflation uses the simple potential $V(\phi) = m^2\phi^2/2$ (where m is indeed the mass of the particles represented by the field). Write down the slow-roll equations for this case and find their solutions with initial conditions $a = a_i$ and $\phi = \phi_i$ at $t = 0$. For what range of ϕ values is the solution inflationary? What condition must be obeyed by ϕ_i to ensure that an expansion of at least 10^{30} takes place?

COSMOLOGY
Problems 7
15 March, 2010

1. Show that the comoving wavenumber corresponding to matter-radiation equality is given by
$$k_{eq} \simeq 0.07\ \mathrm{Mpc}^{-1}\ \Omega_m h^2$$
(up to possible factors of 2π depending on details of the definition of k). Explain why, notwithstanding the fact that this depends on $\Omega_m h^2$, measurements of the matter power spectrum are sensitive to the combination $\Omega_m h$.

2. Another important scale in the distribution of matter in the Universe is the *sound horizon* at recombination. If the speed of sound in the photon-baryon fluid was $c_s = c/\sqrt{3}$, what was its physical size at recombination? What is its comoving size today (i.e., assuming it has just evolved along with the expansion of the Universe since then)? This is the characteristic scale of "baryon acoustic oscillations" in the matter distribution.

3. Show that an isotropic density field has the following relationship between the correlation function, $\xi(x)$ and the power spectrum, $P(k)$:
$$P(k) = 4\pi \int dx\ x^2 \xi(x) \frac{\sin kx}{kx}$$
and find the inverse relation.

4. There is a related relationship between the CMB power spectrum and correlation. We define those as
$$\langle a_{\ell m} a^*_{\ell' m'} \rangle = \delta_{ll'} \delta_{mm'} C_\ell$$
and
$$\left\langle \frac{\Delta T}{T}(\hat{x}) \frac{\Delta T}{T}(\hat{y}) \right\rangle = C(\theta_{xy})$$
where $\cos\theta_{xy} = \hat{x} \cdot \hat{y}$ gives the angle between the directions, and
$$a_{\ell m} = \int d\cos\theta\ d\phi\ Y_{\ell m}(\theta, \phi) \frac{\Delta T}{T}(\hat{x})\ .$$
Find $C(\theta)$ in terms of C_ℓ, and vice versa.